

# Faraday Rotation

Carl W. Akerlof

July 12, 2009

## Introduction

The optical rotation of light by a refractive medium in a magnetic field was first discovered by Michael Faraday<sup>1</sup> in 1845. The effect, like optical rotation induced by organic molecules, is due to *circular birefringence*, the difference in propagation velocities for light of opposite circular polarizations. Since atomic states are weakly perturbed by magnetic fields, the dispersion in the optical refraction indices is of the order of a few parts per million. It is testimony to Faraday's ability as an experimenter to have observed this at all, several decades before the discovery of electric lighting and a century before most of the experimental apparatus we now take for granted. George Fitzgerald<sup>2</sup> (of Lorentz-Fitzgerald contraction fame), using arguments from classical mechanics, provided the connection to the Zeeman effect that had been discovered the previous year, in 1897. In this experiment, you will measure the Faraday effect for two different media and three different wavelengths of light. The results will be analyzed in terms of the optical dispersion of the media and the classical model first described by Fitzgerald.

## Optical Rotation

The geometry of electromagnetic waves is shown below in Figure 1. In (a), a linearly polarized wave is depicted that consists of two equal, in-phase components of the electric field. The polarization lies in a plane defined by the direction of propagation and a line at 45° to the vertical axis. Although the wave in (b) looks quite similar, the 90° shift of phase between the vertical and horizontal electric field components ensures that polarization direction is continually rotating in space and time. There are two naming conventions for circularly polarized light depending on what you consider to vary, time or space. If your location remains fixed and you look at the time dependence of the wave that sweeps past you, the direction of rotation of the electric field defines the handedness. For (b), pointing the thumb of your left hand in the direction of propagation makes your fingers point in the rotation direction of the field – thus, the wave is *left-handed* (LH). On the other hand, if you freeze time and look at the three-dimensional arc traced out by the electric field vector, you see a spiral with the shape of a right-hand screw and is thus called *right circularly polarized* light (RCP). Not surprisingly, the mirror image of (b) is *right-handed* (RH) and *left circularly polarized* (LCP).

A wave linearly polarized along the x-axis can be described by:

$$\mathbf{E} = E_0 \cos(kz - \omega t) \hat{\mathbf{x}}$$

and also be represented as a linear sum of RH and LH states:

$$\mathbf{E} = \frac{1}{2} \mathbf{E}_{RH} + \frac{1}{2} \mathbf{E}_{LH}$$

where

$$\mathbf{E}_{RH} = E_0 \cos(kz - \omega t) \hat{\mathbf{x}} - E_0 \sin(kz - \omega t) \hat{\mathbf{y}}$$

$$\mathbf{E}_{LH} = E_0 \cos(kz - \omega t) \hat{\mathbf{x}} + E_0 \sin(kz - \omega t) \hat{\mathbf{y}}$$

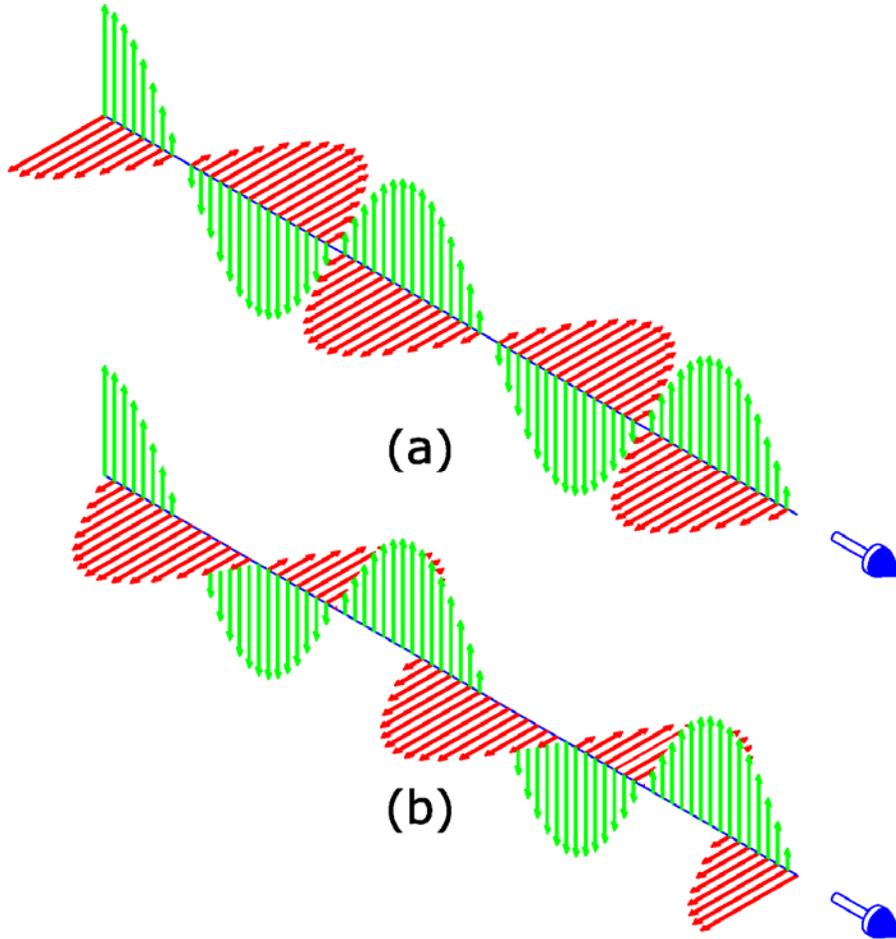


Figure 1. (a) Linearly polarized light – the ray is polarized in a plane at  $45^\circ$  to the vertical; (b) Circularly polarized light – the wave shown is *left-handed* (LH) or equivalently, *right circularly polarized* (RCP).

In a refractive medium,

$$k = \frac{n\omega}{c}$$

where  $n$  is the refractive index which will, in general, be different for left- and right-handed rays. Thus after traversing a distance,  $z$ , in the optical medium, the wave will become:

$$\mathbf{E} = E_0 \left( \cos\left(\frac{\Delta n \pi z}{\lambda}\right) \hat{\mathbf{x}} + \sin\left(\frac{\Delta n \pi z}{\lambda}\right) \hat{\mathbf{y}} \right) \cos\left(\frac{1}{2}(n_L + n_R)kz - \omega t\right)$$

$$\Delta n \equiv n_L - n_R$$

The beam is still linearly polarized but the plane of polarization has been twisted in the clockwise direction (as seen from the ray origin) by an angle:

$$\phi = \frac{\Delta n \pi z}{\lambda}$$

### Connecting the Refractive Index to Atomic Physics

Having described optical rotation in terms of the index of refraction, we must now find a relationship between an applied magnetic field and its effect on the optical properties. Although the Faraday effect was discovered over 160 years ago, textbook explanations<sup>3,4</sup> of the underlying physics are not very detailed and sometimes wrong. The following derivation assumes that the electrons in a dielectric medium behave according to classical mechanics under the combined effect of an external electromagnetic wave and a constant applied magnetic field. This was originally developed<sup>5</sup> in this form by Arnold Sommerfeld in 1949. Predicting the actual direction of rotation of the Faraday effect requires some care in keeping track of the arithmetic signs attached to various terms. In particular,  $e$ , associated with the charge of the electron, is a positive number. The fact that the electron is negative will be accounted for by explicitly inserting the appropriate sign for every term that includes  $e$  as a factor.

The interaction of an electromagnetic field with a dielectric occurs via perturbations to the electrons in atomic or molecular orbitals. Generally, it is virtual transitions from the ground state to the lowest excited state that produces the largest contribution to the refractive index. The energy difference between these states,  $\Delta E = \hbar\omega_0$ , is usually considerably greater than the external E-M wave,  $\hbar\omega$ . Thus, the lifetime of any excited state is exceedingly brief and the energy of the incident wave is undiminished. This sloshing back and forth between adjacent energy levels can be modeled in terms of simple harmonic motion with characteristic frequency,  $\omega_0$ . That assumption will be the starting point for the calculation that follows.

The general plan is to compute atomic electron coordinates for motion induced by the external E-M field and from the associated polarization vector,  $\mathbf{P}$ , calculate the permittivity,  $\epsilon$ , and finally the square of the refractive index. This leads directly to an expression that predicts the magnitude and direction of the Faraday effect with satisfactory accuracy.

The coordinates will be a standard Cartesian grid with z-axis along the direction of light propagation. The applied constant magnetic field,  $\mathbf{B}$ , is oriented in the same direction. Newtonian mechanics gives the following equation of motion

$$\ddot{\mathbf{r}} = -\omega_0^2 - \frac{e}{m}(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

where  $\mathbf{r}$  describes the transverse motion of an electron in the x-y plane. The ratio,  $eB/m$ , is the cyclotron angular rotation frequency, denoted  $\omega_c$ . This leads to the following solutions for  $\mathbf{r}$  in the presence of right-handed or left-handed circularly polarized light:

$$\mathbf{r}_{RH} = -\frac{\frac{e}{m}E_0}{\omega_0^2 - \omega^2 + \omega_c\omega} (\cos(kz - \omega t)\hat{\mathbf{x}} + \sin(kz - \omega t)\hat{\mathbf{y}})$$

$$\mathbf{r}_{LH} = -\frac{\frac{e}{m}E_0}{\omega_0^2 - \omega^2 - \omega_c\omega} (\cos(kz - \omega t)\hat{\mathbf{x}} - \sin(kz - \omega t)\hat{\mathbf{y}})$$

At this point, we can calculate the induced polarization of the medium via:

$$\mathbf{P} = -Ne\mathbf{r}$$

where  $N$  is the atomic number density. Furthermore, it is convenient to parametrize  $N$  in terms of the plasma frequency,

$$\omega_p^2 = \frac{Ne^2}{\epsilon_0 m}$$

Thus, we have the following expressions for the polarization vectors,  $\mathbf{P}_{RH}$  and  $\mathbf{P}_{LH}$ :

$$\mathbf{P}_{RH} = \frac{\epsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 + \omega_c\omega} \mathbf{E}_{RH} ; \quad \mathbf{P}_{LH} = \frac{\epsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 - \omega_c\omega} \mathbf{E}_{LH}$$

Since:

$$\epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

and for most dielectrics,  $\mu \cong \mu_0$ , so that:

$$n^2 \cong \epsilon/\epsilon_0$$

It follows that the refractive indices for the two circular polarization states are:

$$n_{RH}^2 = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + \omega_c\omega} ; \quad n_{LH}^2 = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \omega_c\omega}$$

Noting that:

$$\omega^2 \mp \omega_c \omega \cong \left( \omega \mp \frac{1}{2} \omega_c \right)^2,$$

we obtain the circular birefringence:

$$\begin{aligned} n_{LH} - n_{RH} &= \sqrt{n^2 \left( \omega + \frac{1}{2} \omega_c \right)} - \sqrt{n^2 \left( \omega - \frac{1}{2} \omega_c \right)} \\ &\cong \omega_c \frac{dn}{d\omega} \end{aligned}$$

This leads directly to an expression for the Faraday rotation angle introduced earlier:

$$\begin{aligned} \phi &= \omega_c \frac{dn}{d\omega} \frac{\pi z}{\lambda} = \frac{\omega_c \omega z}{2c} \frac{dn}{d\omega} = \frac{e}{2mc} \omega \frac{dn}{d\omega} \int \mathbf{B} \cdot d\mathbf{z} \\ &\equiv \mathcal{V} \int \mathbf{B} \cdot d\mathbf{z} \end{aligned}$$

This formula was first derived<sup>6</sup> by Henri Becquerel (of radioactivity fame) in 1897 and bears his name. The term denoted by  $\mathcal{V}$  is called the Verdet constant<sup>7</sup> after Pierre Verdet who was first to note the explicit linear dependence on the integral,  $\int \mathbf{B} \cdot d\mathbf{z}$ . The most important aspect of this result is the link between the frequency of the incident radiation,  $\omega$ , and the cyclotron frequency perturbation,  $\omega_c$ . Thus, the dependence of the Faraday rotation angle on  $dn/d\omega$  is likely to survive, even in cases where  $n$  is described by more complicated physical models. As noted<sup>8</sup> by Robert Serber (who became much more famous for his participation<sup>9</sup> in the Manhattan Project), the expression for  $\mathcal{V}$  is modified by the restrictions of atomic transition g-factors, leading to:

$$\mathcal{V} = \alpha \frac{e}{2mc} \omega \frac{dn}{d\omega}; \quad 0 \leq \alpha \leq 1$$

In the experiment that follows, you should explore the following characteristics of Faraday rotation to test this microscopic model:

1. Does the direction of rotation depend on the direction of the applied magnetic field?
2. Does the magnitude of the Faraday rotation depend linearly on  $\int \mathbf{B} \cdot d\mathbf{z}$ ?
3. Using the Sellmeier coefficients to compute the dispersion of the refractive index, determine if the frequency (or wavelength) behavior of Faraday rotation is experimentally observed.

4. Compute the value for  $\alpha$  given above by comparing the ratio of observed to computed values of the rotation angle,  $\varphi$ . Comment on the extent to which  $\alpha$  does or does not vary as a function of wavelength or glass sample.

### Experimental Apparatus

The experimental apparatus for this experiment is shown in Figure 2 below. The basic arrangement is quite simple: a beam of polarized light from a laser passes through a transparent sample immersed in a longitudinal magnetic field before passing through a second rotatable polarizing filter and absorbed finally by a silicon photodiode detector. The intensity of this remaining beam is determined by measuring the photodiode current across a 10 K $\Omega$  series resistor. For highest sensitivity, use the millivolt scale on the Agilent U1252A multimeter. The direction of polarization is measured by rotating the polarizer to minimize the photo signal and reading the relative orientation from the scale printed on the rotary mount (for more details, see discussion of use of vernier scales).

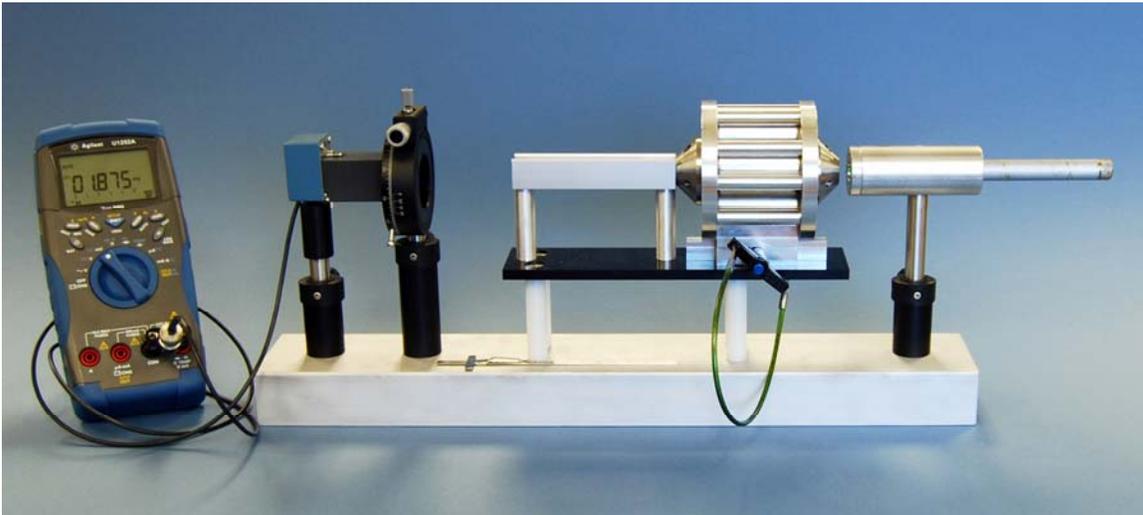


Figure 2. Faraday rotation apparatus. From right to left; laser pointer light source, magnet assembly and sample holder, rotating polarizer mount, silicon photodiode detector, digital multimeter.

The magnetic field is produced by a ring of five permanent rare earth (NdFeB) magnets housed in an aluminum enclosure and equipped with steel pole tips. The field of these magnets cannot be varied but the samples can be positioned at different locations to vary the integral of the field over the sample lengths. As shown in Figure 3, the central field is greater than 3 kilogauss and field integrals of the order of 0.012 Tesla-meters are obtainable for the samples of interest. Some care should be exercised with this device – **Do Not Bring Steel Objects Anywhere Nearby!** Sample positions should be determined with respect to the center of the magnet array. Since the overall distance between the two steel pole tips is 4.005”, such measurements can be referenced from the ends as well.

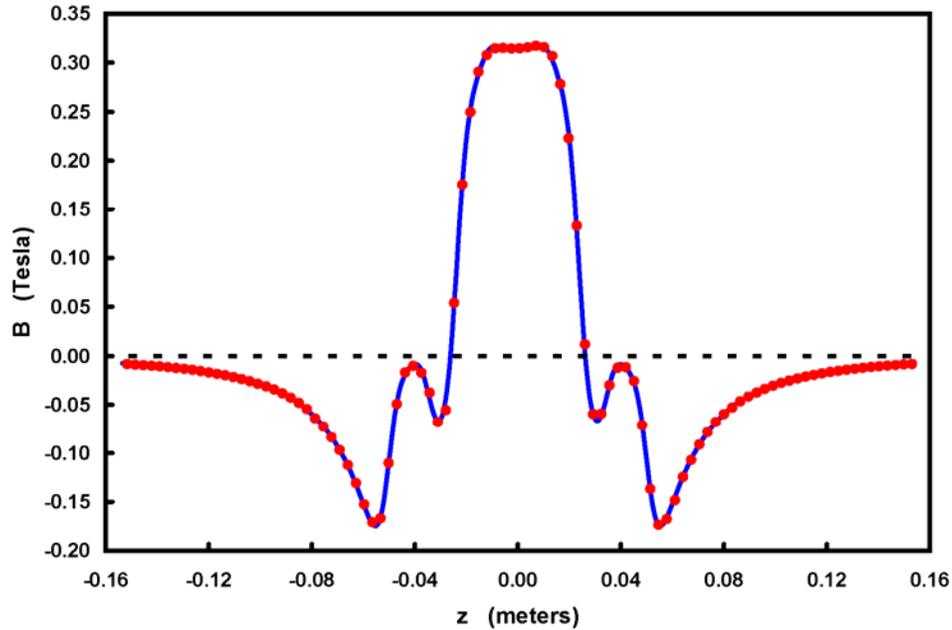


Figure 3. Longitudinal magnetic field as a function of distance along the symmetry axis. The red points show the measured data; the blue line is a B-spline fit, symmetric about  $z = 0$ .

The three lasers available for use in this experiment are shown in Figure 4. They provide monochromatic beams at wavelengths of 473, 532 and 650 nm. For the blue laser, an orange Velcro strap has been provided to keep the *On* switch depressed. Please remove it as soon as possible after use since the battery for this gadget is a bit pricey.

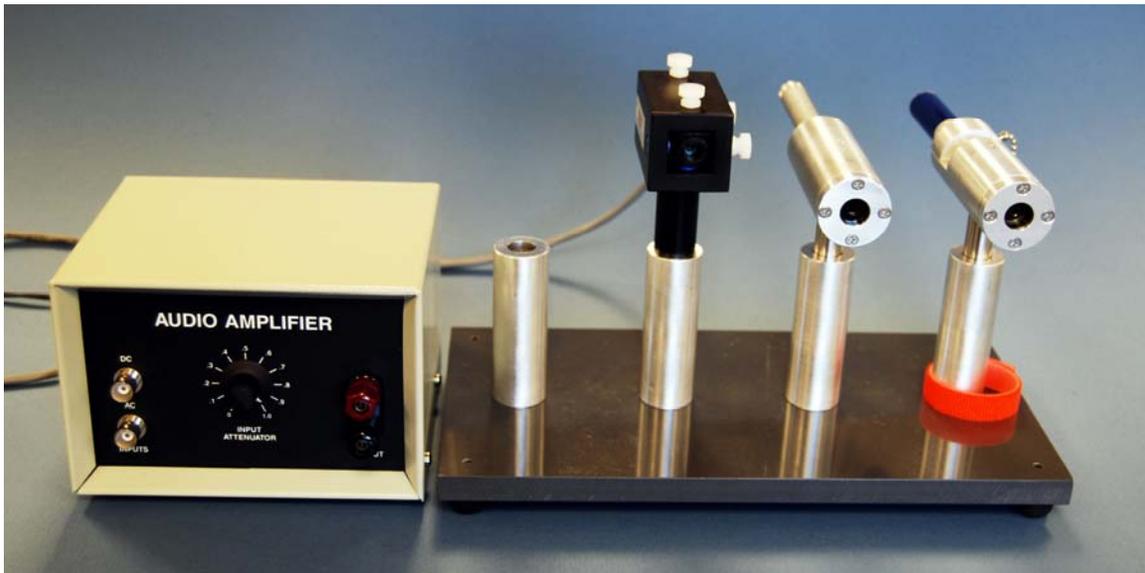


Figure 4. Array of laser light sources: From right to left; blue ( $\lambda = 473$  nm), green ( $\lambda = 532$  nm), red ( $\lambda = 650$  nm). The red laser requires the power supply on the left; the other two are battery-powered.

Two transparent samples are available: BK-7 (3" long hexagonal prism) and SF-59 (4" long yellowish cylindrical rod). These materials are optical glasses manufactured by the Schott Corporation. Optical design requires extremely accurate knowledge of the refractive index across the visual spectrum. Thus, the indices of these materials are characterized to one part in 200,000. BK-7 is the cheapest and most common optical glass with relatively low dispersion and moderate index ( $n_d \sim 1.52$ ) while SF-59, with a very high Pb content, has a high dispersion and high index ( $n_d \sim 1.95$ ).

The only adjustments required for this experiment are laser beam collimations through the center of the samples. Figure 5 below shows the green laser beam correctly exiting the SF-59 lead glass sample.

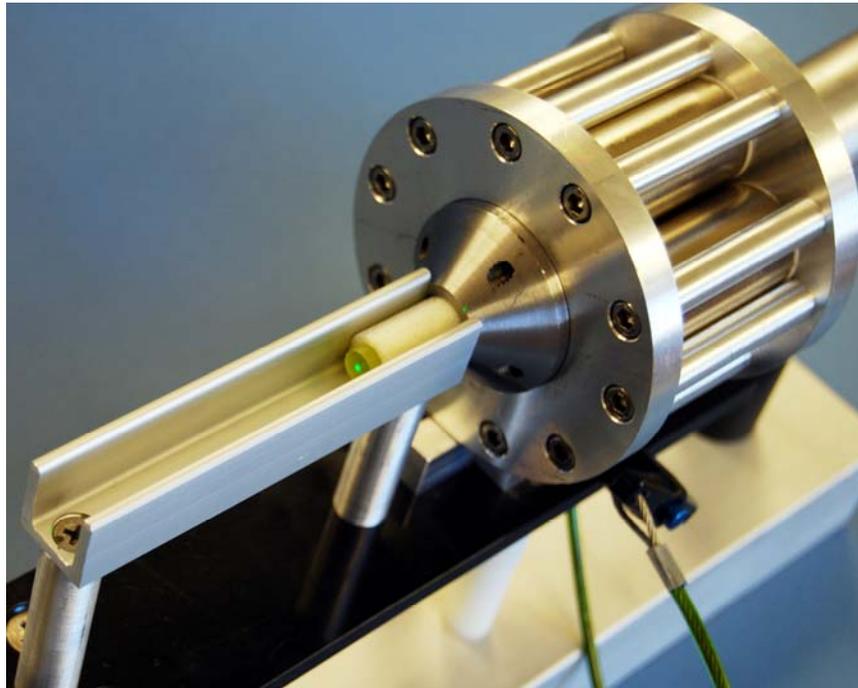


Figure 5. Properly collimated green laser beam exiting the SF-59 glass sample.

### The Use of Vernier Scales

In a digital age, it is easy to assume that measurements will always be presented as strings of numeric digits. However, in this experiment, the rotating polarizer mount employs an analog scale invented by the French mathematician Pierre Vernier in 1631. By clever geometry, the vernier scale achieves accuracies that are far better than the primary rulings on the instrument. The basic principle is shown in Figure 6. The red vernier scale interpolates the upper scale, providing an extra digit of precision. Note that for the polarizer mount used in this experiment, the coarse scale is ruled in degrees and the vernier scale is ruled in increments of 5 arc-minutes.

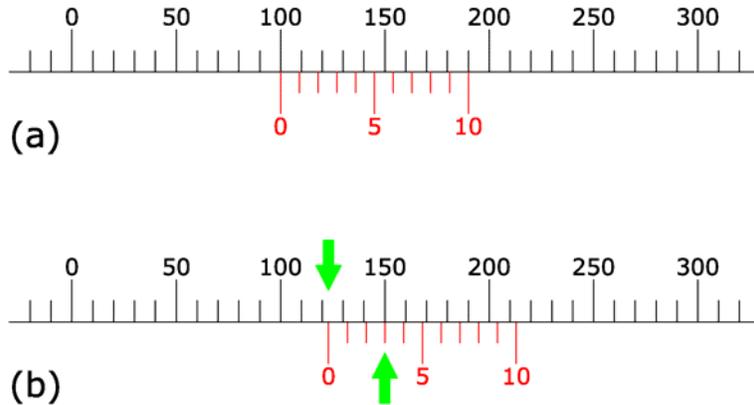


Figure 6. Two examples of vernier scales. In (a), the **vernier scale** origin points directly at 100. In (b), the origin of the **vernier scale** lies somewhere between 120 and 130 (upper green arrow). Looking at the match between the upper and lower rulings, the lines that intersect most closely correspond to “3” on the **vernier scale** (lower green arrow). Thus, the appropriate reading would be “123”.

### Experimental Procedure

The overall procedure is designed to explore the behavior of the Faraday effect as a function of wavelength and magnetic field. For each of three wavelengths, you should establish the extinction point for the analyzing polarizer with no sample in place. You may find it easier to find the exact extinction angle by finding roughly equivalent transmissions a degree or so on either side of the minimum and averaging the two. Obtain data for the three wavelengths with the glass samples centered in the magnet assembly. Make measurements for both directions of the magnetic field. This is accomplished by removing the magnet locking pin, lifting up the magnet assembly and reversing its direction. You can determine your measurement errors by comparing the average of the rotation angles corresponding to the two field directions with the extinction angle for no sample whatsoever. Ascertain the actual direction of the magnetic field inside the bore by using the longitudinal field Hall probe and compare with the field from a current loop whose field direction you can confidently predict. Finally, explore the dependence of the Faraday rotation on the magnetic field integral using the green laser (because it's quite stable) and the SF-59 sample (because the effect is larger). Position the sample at various distances from the magnet center to explore magnetic field integrals from +12 to -6 millitesla-meters (see Figure 8 to choose appropriate offset distances).

### Data Analysis

The data you have taken should be analyzed in terms of the model developed earlier. To accomplish this, you will need values for  $\int Bdz$  and  $dn/d\omega$  corresponding to the appropriate experimental conditions. The values for the magnetic field were shown

earlier in Figure 3. To simplify further computation, the data has been fitted to a smooth curve (cubic B-splines) and integrated to produce the curve shown in Figure 7.

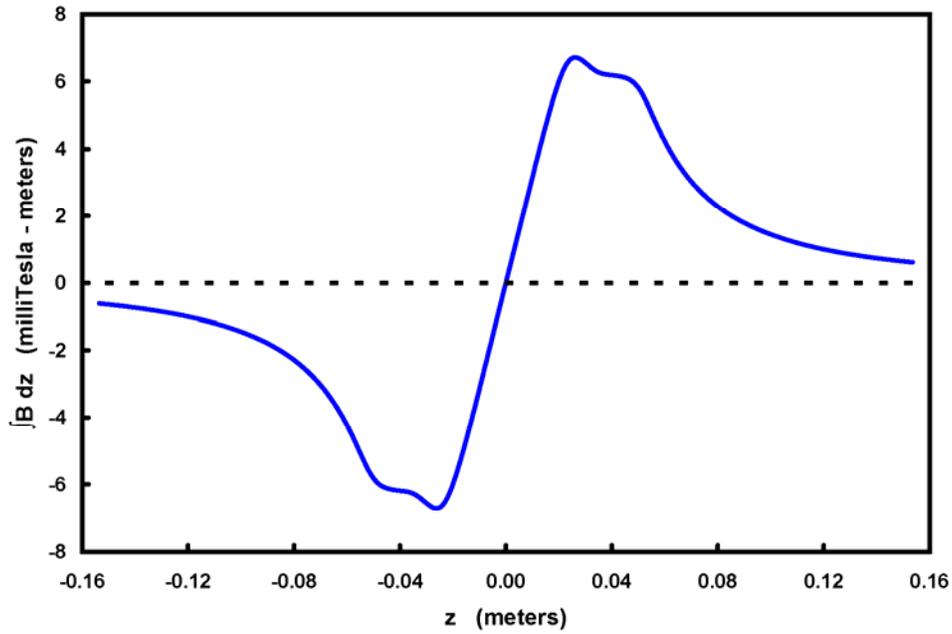


Figure 7.  $\int_0^z B(z') dz'$  computed from a B-spline fit to the magnetic field data.

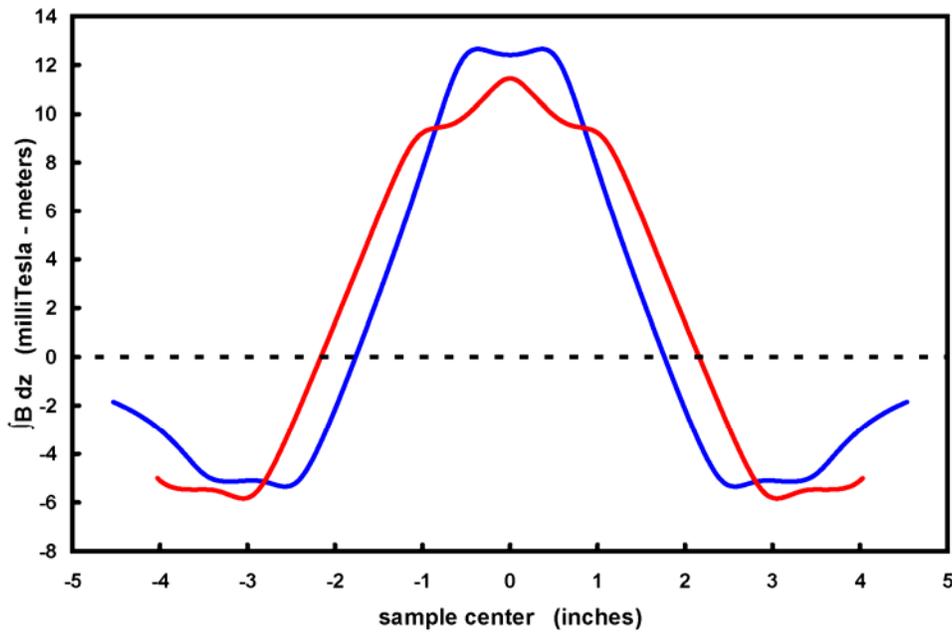


Figure 8.  $\int_{z-\frac{\ell}{2}}^{z+\frac{\ell}{2}} B(z') dz'$  as a function of center position for 3" long (blue) and 4" long (red) samples.

To compute the effective field integrals over the two optical glass samples, the integrals between sample endpoints were differenced to produce the curves shown in Figure 8. These values can be obtained for arbitrary sample length and position from the Excel spreadsheet, *Faraday\_field.xls*, available on the Physics 441/442 Web site.

$\lambda_F$	486.13	nm
$\lambda_d$	587.56	nm
$\lambda_C$	656.27	nm

Table I. Wavelengths used to define the Abbe number for refractive media.

The optical properties of glasses used for high quality lenses are generally characterized by two parameters: the index of refraction,  $n_d$ , at a wavelength of 587.56 nm and the Abbe number,  $V_d$ , defined by:

$$V_d \equiv \frac{n_d - 1}{n_F - n_C}$$

The Abbe number (not to be confused with the Verdet constant,  $\mathcal{V}$ ) indicates the dispersiveness of the medium – a low value means that the refractive index varies considerably with wavelength. With some simplifying assumptions, the Abbe number can be used to make an estimate of  $\omega \cdot dn/d\omega$ , a quantity needed to predict the magnitude of the Faraday effect. To a crude approximation, the refractive index,  $n$ , is described by:

$$n \cong a + b/\lambda^2$$

where the coefficients,  $a$  and  $b$ , are related to  $n_d$  and  $V_d$  by:

$$b = \frac{\lambda_F^2 \lambda_C^2}{\lambda_C^2 - \lambda_F^2} \frac{n_d - 1}{V_d}$$

$$a = n_d - \frac{b}{\lambda_d^2}$$

This leads to the following estimate for  $dn/d\lambda$ :

$$\frac{dn}{d\lambda} = -\frac{2b}{\lambda^3}$$

For this experiment, we would like to obtain the values for  $dn/d\lambda$  to considerably greater precision. Optical designers need to know the wavelength dependence of the indices of refraction of glasses to very high accuracies in order to minimize chromatic aberrations that would otherwise ruin the performance of high quality lenses. The standard representation is called the Sellmeier formula:

$$n^2(\lambda) = 1 + \frac{B_1\lambda^2}{\lambda^2 - C_1} + \frac{B_2\lambda^2}{\lambda^2 - C_2} + \frac{B_3\lambda^2}{\lambda^2 - C_3}$$

Note that the equation computes the *square* of the index of refraction. For the glass samples used in this experiment, the parameters are given in Table 2 below. Using this representation will allow you to compute  $dn/d\lambda$  (and thus  $dn/d\omega$ ) far more accurately than possible with the Abbe number.

	Schott SF-59	Schott BK-7
$B_1$	2.05775824	1.03961212
$B_2$	$5.28644661 \times 10^{-1}$	$2.31792344 \times 10^{-1}$
$B_3$	1.00043574	1.01046945
$C_1$	$1.67799599 \times 10^{-2}$	$6.00069867 \times 10^{-3}$
$C_2$	$6.55335804 \times 10^{-2}$	$2.00179144 \times 10^{-2}$
$C_3$	$1.10002549 \times 10^2$	$1.03560653 \times 10^2$
$n_d$	1.95250	1.51680
$V_d$	20.36	64.17

Table II. Sellmeier coefficients and glass characteristics for Schott SF-59 and BK-7 glasses. The coefficients are defined for wavelengths specified in microns.

### Acknowledgements

The author gratefully thanks Paul Berman for a number of discussions of the classical and quantum mechanical descriptions of Faraday optical rotation.

### References

1. Michael Faraday, *Experimental Researches in Electricity: On the magnetization of light and the illumination of magnetic lines of force*, Philosophical Transactions of the Royal Society of London **136**, 1-20 (1846).
2. George Fitzgerald, Note on the Connection between the Faraday Rotation of Plane of Polarisation and the Zeeman Change of Frequency of Light Vibrations in a Magnetic Field, Proceedings of the Royal Society of London **63**, 31-35 (1898).
3. K. D. Möller, *Optics*, pp. 299-305, University Science Books (1988).
4. Robert Guenther, *Modern Optics*, pp. 590-596, John Wiley & Sons (1990).

5. Arnold Sommerfeld, translated by Otto Laporte & Peter A. Moldauer, *Lectures on theoretical physics: volume IV - Optics*, pp. 101-106, Academic Press (1964).
6. Henri Becquerel, *Sur une interprétation applicable au phénomène de Faraday et au phénomène de Zeeman*, Comptes rendus hebdomadaires des séances de l'Académie des sciences **125**, 679-685 (1897).
7. Émile Verdet, Note sur les propriétés optiques des corps transparents soumis à l'action du magnétisme, Comptes rendus hebdomadaires des séances de l'Académie des sciences **43**, 529-532 (1856).
8. Robert Serber, *The Theory of the Faraday Effect in Molecules*, Phys. Rev. **41**, 489-506 (1932).
9. Robert Serber, *The Los Alamos Primer: The First Lectures on How To Build an Atomic Bomb*, University of California Press (1992).