The Planck blackbody spectral distribution and measurement of the solar photosphere temperature

Carl W. Akerlof
July 27, 2009

Introduction

The characteristics of black body radiation are manifest in a variety of different contexts for both science and technology. At a time when energy sources and consumption have become issues of national concern, it is useful to look at some basic limits determined by quantum mechanics and thermodynamics. The experiment outlined here is an extension to the one I developed some years ago for Physics 341. The principle goal is to estimate the surface temperature of the Sun by measuring the photon intensity spectral distribution. This technique is of enormous importance for stellar astrophysics. The fact that one can probe the characteristics of physical systems at almost unimaginable distances is one of the triumphs of modern science. With careful measurements and good mathematical data reduction techniques, good results can be obtained with very few assumptions.

Blackbody radiation and the Stefan-Boltzmann Law

The first origins of quantum mechanics arose from the failure of classical methods to explain the spectral distribution of light from hot objects. The experimental model system is a hot cavity such as an oven with walls at a uniform temperature T. A small hole is cut into an oven wall to allow a small fraction of the electromagnetic energy to escape. Such light is called *blackbody radiation*.

Statistical mechanics plus classical electromagnetism predicted a spectral distribution called the Rayleigh-Jeans law that can be written as:

$$\frac{dI}{df} = \frac{2\pi f^2}{c^2} kT \tag{1}$$

where I is the intensity (power per unit area) of light emitted per unit frequency at frequency f by an object at temperature T. Although the derivation of this formula is beyond the scope of this course, the form can be easily explained. In classical statistical mechanics, every possible degree of freedom should acquire an average energy of ½ kT. For example, a monoatomic gas molecule has $(3/2)kT$ energy because it can move in three independent directions. The number of independent modes of oscillation in a cavity scales like f^2 for reasons similar to the relationship of the area of a sphere to its radius. The total intensity at a given frequency is just the product of the number of modes available and the average energy in each one. This is the physical content of Equation 1.

From a practical point of view, Equation 1 is a disaster. As the frequency, f, becomes large, the predicted intensity increases without limit, even for objects at modest temperature.

This is called "the ultraviolet catastrophe". On the other hand, at low frequencies, the formula gave accurate predictions of experimental results, indicating that at least some aspects were basically correct.

The problem was solved in two steps. Since Equation 1 leads to an infinite radiation rate, another approach must be found to compute the total energy emitted by a hot object. Ludwig Boltzmann found a thermodynamic argument to show that the total intensity is given by:

$$I = \sigma T^4 \tag{2}$$

where σ is a constant of nature and T is the absolute temperature. This agreed with earlier experimental measurements by Josef Stefan. Equation 2 is called the Stefan-Boltzmann Law - the unusually rapid increase in radiation with temperature is a consequence of the massless nature of photons, the carriers of electromagnetic energy (the same kind of behavior governs phonons, the quanta of acoustic energy in solids and liquids).

Although this relationship describes the total energy emitted, it does not predict the spectral distribution. That step was taken by Max Planck who postulated that electromagnetic energy was emitted in discrete units or quanta, each with energy given by hf where h is the Planck constant, 6.6261 x 10^{-34} joule-second (see Table I). For photon energies with $hf > kT$, it would no longer be possible to populate each mode with kT average energy since a fraction of hf is no longer allowed. The consequence is an additional factor of

$$\frac{hf/kT}{e^{hf/kT}-1}$$

that reduces the spectral distribution given by Equation 1. This factor approaches unity for small f, preserving the long wavelength Rayleigh-Jeans behavior but squelches the ultra-violet divergence. The result is the Planck spectral distribution:

$$\frac{dI}{df} = \frac{2\pi h f^3}{c^2}\frac{1}{e^{hf/kT}-1} \tag{3}$$

By integrating this equation over all frequencies, one can recover Equation 2 and, in addition, find that the constant, σ, is explicitly given by:

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} = 5.67\times 10^{-8} W/m^2 \cdot K^4 \tag{4}$$

The integration of Equation 3 is accomplished by rewriting (3) as:

$$\frac{dI}{dx} = \frac{kT}{h}\,\frac{dI}{df} = \frac{2\pi k^4 T^4}{h^3c^2}\,\frac{x^3}{e^x-1} \tag{5}$$

where $x = \dfrac{hf}{kT}$.

Since $\int_0^\infty \frac{x^3 dx}{e^x - 1} = 6\zeta(4) = \frac{\pi^4}{15}$, Equation 4 follows immediately. The number of photons emitted per unit of time, area and frequency is given by:

$$\frac{d\dot{N}}{df} = \frac{1}{hf}\frac{dI}{df} = \frac{2\pi f^2}{c^2} \frac{1}{e^{\frac{hf}{kT}} - 1} \tag{6}$$

This can be also integrated in a similar fashion to yield:

$$\dot{N} = \frac{2\pi k^3 T^3}{h^3 c^2} \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{2\pi k^3 T^3}{h^3 c^2} \; 2\zeta(3) = 1.52046630 \times 10^{15} T^3 \tag{7}$$

Dividing I by $\dot{N}$, we find that the mean photon energy is 2.70117803 kT. A long list of similar characteristic parameters of the Planck distribution is provided in Appendix A.

c	299792458 m/s
e	$1.602176487 \times 10^{-19}$ C
h	$6.62606896 \times 10^{-34}$ J s
k	$1.3806504 \times 10^{-23}$ J °K^{-1}
σ	$5.67040047 \times 10^{-23}$ W m^{-2} °K^{-4}
$R_\odot$	695990 km
AU	$1.49597870691 \times 10^{8}$ km

Table I. Useful constants and parameters.

Experimental strategy

The basic goal of this experiment is the determination of the solar surface temperature from the relative intensities of the spectrum sampled at a number of wavelengths from 450 to 950 nm. The sampling is determined by a set of broadband interference filters that modulate the light intensity measured by the photocurrent in a reverse biased silicon diode. If the detailed characteristics of the filters and photodiode were all initially well known, a single set of measurements of sunlight through the filter set would suffice. In the absence of such information, the light from a tungsten lamp at different temperatures will be used to establish the appropriate calibrations. In addition, sunlight is also reddened by the atmosphere which differentially absorbs the blue end of the spectrum. This effect can be corrected by measuring the spectral intensities as a function of zenith angle. Finally, least squares techniques determine the solar temperature by modeling the data with the Planck spectral distribution function.

Measuring light intensity

The basic device for measuring light intensity in these experiments is a silicon diode wired so that the internal junction is reverse biased. A silicon diode is the most elementary semiconductor structure one can construct. Schematically it looks like the sketch shown in

Figure 1. In the upper region of the diode, electric current is transported by positive charge carriers called "holes"; in the lower region, the current carriers are negatively charged electrons. If this device is "reverse-biased" so that the upper electrode is more negative than the lower one, the charge carriers will separate as shown in Figure 2. In this case, the positive "holes" will be attracted to the top and the electrons towards the bottom, leaving a "depletion layer" in the middle. Since almost no free electrical carriers are present there, the current through the diode is effectively reduced to zero. With these circumstances, an incident light beam will produce electron-hole pairs in the depletion region and allow an electric current to flow once more. The total charge conducted will be directly proportional to the **number** of incident photons absorbed and, for a well-designed photodiode, that ratio is of the order of unity. The photodiodes for these experiments are mounted in small blue boxes and wired as shown in Figure 3. Measuring the photocurrent is fairly easy if the current is large enough. The Agilent U1252A multimeter will operate as an ammeter down to about 0.01 μA (10^{-8} A).

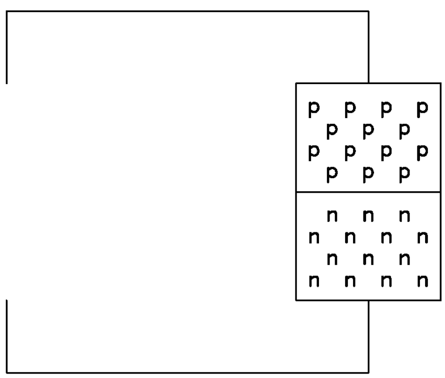


Figure 1. Schematic view of a silicon *pn* diode.

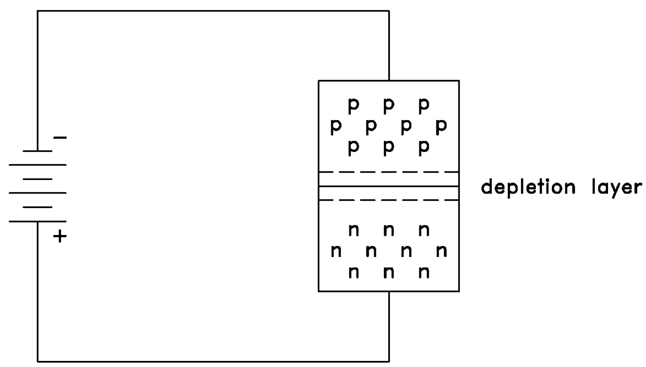
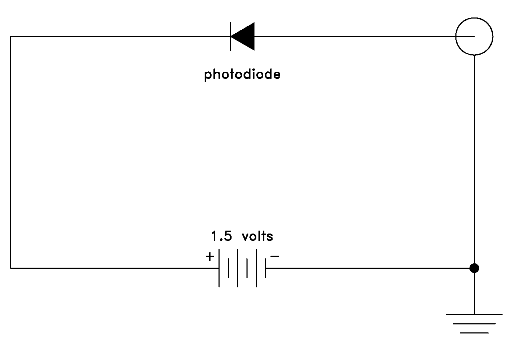


Figure 2. By reverse biasing a *pn* diode, a charge-free region is created at the *pn* boundary called the depletion layer.

A note of caution: although the voltage output of a photodiode is related to the input light intensity, the relationship is far from linear. For an open circuit (i.e. no current flow), the voltage is practically constant, independent of intensity as long as the light is bright enough to overwhelm the intrinsic electron-hole recombination rate. The terminal voltage under these circumstances is the bandgap potential which is 1.11 volts for silicon. The explanation for similar effects in metals was the scientific contribution for which Einstein was awarded the Nobel Prize in 1921. **Do not use the multimeter in voltage mode to measure light intensity!**

Figure 3. Wiring diagram for the photodiode box.



Tungsten light source

The calibration light source for this experiment is a #1446 miniature screw base tungsten filament lamp that operates at a nominal 12 volts and a current of 0.2 amperes. It was chosen for its relatively high resistance, easy replaceability and low cost. By operating with a DC power supply, the filament temperature is held constant, important for reliable measurements. For estimating temperatures, the power consumption of the lamp must be carefully measured. This requires knowing both the electric current passing through the tungsten filament and the voltage across it. The resistance of the lamp can be used to estimate the filament temperature since the resistivity increases in a well-known way as the metal gets hotter as tabulated in Table II. The Hewlett-Packard E3632A power supply is convenient for these measurements since the output voltage and current are monitored and displayed. To operate the supply, turn on the power with the *Power On/Off* pushbutton, enable the supply with the *Output On/Off* button and select the voltage range with the *Range 30V, 4 A* button. Do not exceed 20 volts across the lamp when used as a filter calibration source.

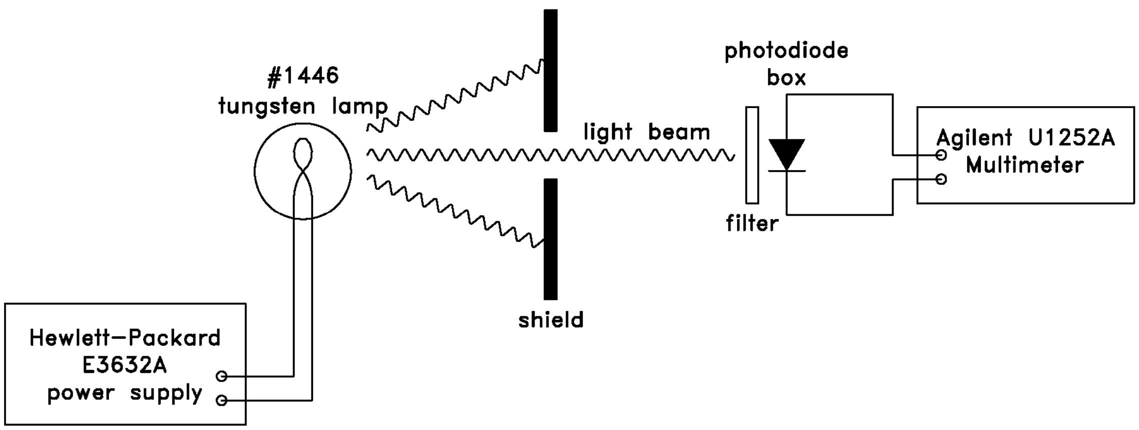


Figure 4. Arrangement of lamp, shield, filter and photodiode sensor

Filter calibration procedure

Set up the tungsten lamp, light shield, filters, power supply and Agilent multimeter as shown in Figure 4. The basic plan is to measure the photocurrents at six temperatures with six different interference filters. The filament temperatures will be determined by calculating the operating resistance relative to room temperature and applying corrections imposed by the Stefan-Boltzmann Law. This means that the room temperature resistance of the tungsten lamp must be measured accurately. An Agilent 34401A multimeter is available for this purpose. Since the filters transmit a fairly small fraction of the total spectrum, it is a good idea to keep as much stray light from the photodiode as possible. For this reason, place a large aluminum screen with the hole centered on the light path between filament and diode. For each filter, the optical bandpass center wavelength is marked on the upper edge of the holder. Operate the lamp at six different voltages: 7.5, 10.0, 12.5, 15.0, 17.5 and 20.0. The later three exceed the design limits for the lamp so try to keep these exposures as short as possible. For each voltage/filament temperature, make note of the power supply voltage and current. These will be needed to derive the filament temperature. You should also measure the lead wire resistances with the Agilent 34401A to correct for power loss in the connecting wires.

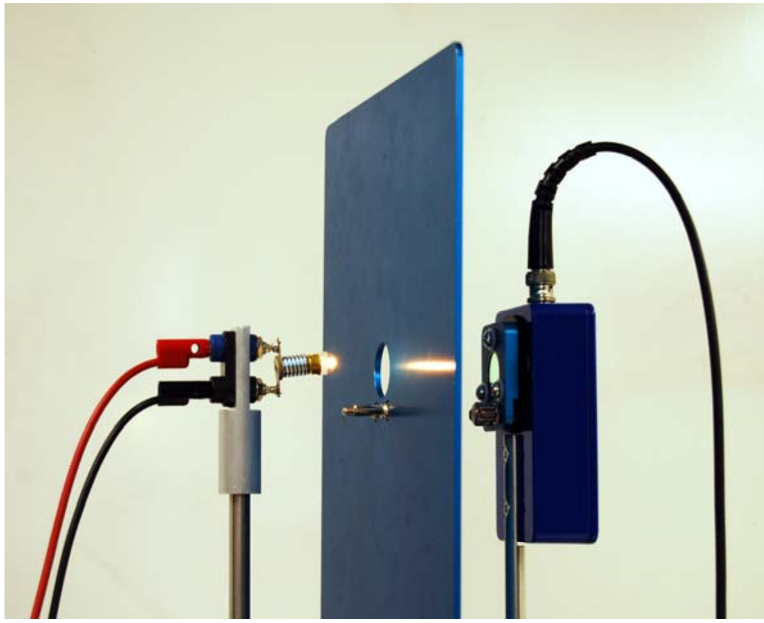


Figure 5. Arrangement of tungsten lamp, light shield, filter and photocell for calibration measurements.

The next step is to estimate the actual operating temperature for each of the six voltages. By this time, you should have found the total resistance for the two lamp current wires and the room temperature value for the filament itself. From the voltage and current provided by the power supply, corrected for lead losses, you can then determine the filament resistance at the operating temperature and from its ratio relative to 293º K, find the actual temperature. An Excel file, *solar_template.xls*, is available to perform the appropriate interpolation from a table of resistance values.

To model the set of photocurrents measured for the various temperatures and wavelengths, we need to make a few assumptions and then define a set of parameters to be determined by the data. The most critical assumption is that the light emitting region of the filament remains constant, independent of temperature. This is probably not strictly correct but is at least a reasonable approximation. The photocurrent observed with each filter will then be the product of the Planck spectral distribution at that wavelength or frequency, the filter relative transmission probability and an overall amplitude related to the area of the filament and beam geometry. Since the filter transmissions are relative, one of them must be held to an arbitrary value such as unity. That means that there are six independent parameters to be determined by applying the least squares method to minimizing the difference between observed and fitted values. Since the data range over quite a large span of values, it is best to define the least squares sum in terms of the relative error of each pair of observation and fit. I used the Excel *Solver* tool to accomplish this function – for other mathematical packages, you're on your own.

Checking the Stefan-Boltzmann Law

You probably have already verified the Stefan-Boltzmann Law in Physics 341. However, since we are willing to sacrifice a light bulb to the cause of science, you can explore a somewhat wider range of temperatures with the apparatus at hand. In this case, vary the power supply voltage from 6 to 30 volts in steps of one volt and measure the associated current. Compute the lamp filament temperature from the ratio of resistance relative to room temperature. Fit the lamp power to a power law in temperature and compare the exponent with the Stefan-Boltzmann value of 4. The obvious consequence of pushing the power output of the lamp way beyond its design is irreversible blackening of the glass bulb due to evaporation of the filament, ultimately leading to failure. Thus, once a lamp is exposed to such extreme treatment, it should not be used for filter calibrations again.

Measurements of the solar photospheric temperature

There are two aspects of the solar radiation that can be easily measured: the spectral distribution and the light intensity. Both of these can be used to estimate the solar surface temperature. To perform the required measurements, you will need all the same equipment used for calibration but without the power supply and the tungsten lamp. It is best to perform at least one measurement near solar transit. The terrestrial coordinates for the Block M on the Diag are 42º 16.621′ N, 83º 44.292′ W, 280 m altitude. This means that solar transit occurs approximately at 12:35 pm EST or 1:35 pm EDT. It is easy to check this estimate: the campus buildings are oriented due north-south and east-west so you can verify if the solar shadows line up accordingly at the predicted time. A convenient place to set up the apparatus is the north-west corner of the courtyard between Randall and West Hall. The bench surrounding the skylight can be used as a table and you can check the zenith angle value by measuring the shadow cast by the surrounding protective railing. The proper aim towards the Sun can be obtained by ensuring that sunlight passes cleanly through the hollow optical bench rails. Measure the photocurrent for all six filters, making sure to also obtain associated sky backgrounds by aiming the system five or ten degrees off-source so that direct sunlight does not illuminate the photocell.

The atmosphere poses one problem that is not an issue in the laboratory, the absorption and scattering of light. This is wavelength dependent, preferentially depriving direct sunlight of photons at the blue end of the spectrum. Astronomers refer to this as *atmospheric extinction* and measure it in the rather quaint units of magnitude. The transmission through a medium with m magnitudes of extinction is $10^{-0.4\,m}$ – thus, 5 magnitudes will reduce the flux by a factor of 100. For observing celestial objects, the extinction will increase proportional to the air mass through which the light traverses so that the effective extinction scales like $m_0/cos(\theta)$ where m_0 corresponds to the extinction at zenith. If the weather is suitable, not something that one can count on in Ann Arbor, it would be nice to measure the extinction directly by taking at least three separate sets of measurements that span a factor of two in air mass, ie. the ratio of values for $cos(\theta_{zenith})$. The solar zenith angles can be calculated via the interactive Web program at http://www.nrel.gov/midc/solpos/spa.html. If the weather does not cooperate, use the estimated values provided in Table V.

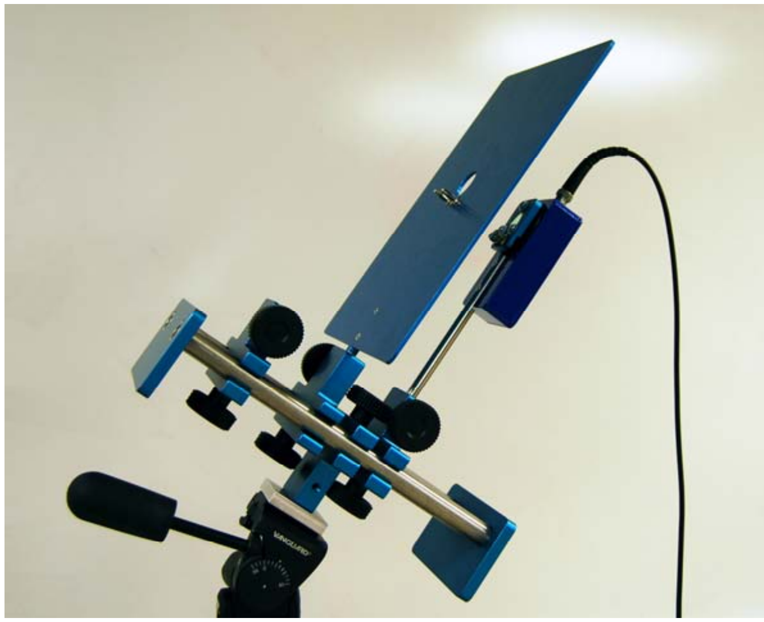


Figure 6. Light shield, filter and photodetector arranged for solar radiometry measurements.

To find the solar temperature, model the photocurrent dependence on wavelength and temperature using the Planck spectral distribution, the filter transmissions found earlier, and the atmospheric extinctions discussed above. There are two free parameters: the solar temperature and an overall normalization amplitude. Use *Solver* or some similar optimizer to minimize the relative error between measurement and model.

A second independent method is to estimate the solar temperature directly from the measured photon flux. In this case, you need to assume the solar surface area, the distance to the Sun and the parameters for the detection apparatus, given in Tables III and IV. Try this for the λ = 650 nm observation since this wavelength is least affected by various uncertainties in atmospheric attenuation. Note that the photodiode is exactly 1 cm^2 in area. (Warning: note that the photocurrent is proportional to *photon number*, not *photon energy*.)

Economic implications of the Planck distribution

Assume that all incandescent tungsten lamps operate at the temperatures encountered while calibrating the filters for this experiment. Since incandescent lamps must have a reasonable lifetime before burnout, T $\simeq$ 2350° K is a typical compromise for an operating point. For human vision, the useful photon wavelengths range between 450 and 650 nm. Find the fraction of energy within this segment of the blackbody spectrum relative to the total emitted electromagnetic radiation. (Hint: Numerically integrate Equation 3 using the parameterization described by Equation 5.) What are alternative electrical lighting technologies and why are they better even if more expensive to manufacture?

Solar cells generate electrical energy that can be estimated by the product of the cell bandgap energy and the number of photons absorbed with quantum energies above the bandgap. For a specific temperature of the blackbody radiation, the output will be optimized when the bandgap is 2.166 kT. Find the optimal bandgap energy and the maximum output energy efficiency of a solar cell under these conditions, assuming a solar surface temperature of 5778° K. Remember that the maximum power that can be usefully delivered by any kind of power source, solar or otherwise, is one half the energy generated (the other half must be dissipated internally). For the mathematically inquisitive, find the appropriate equation that determines the optimal bandgap voltage quoted above. The yearly electrical consumption in the United States is in the neighborhood of 1.5×10^{19} Joules. How many square kilometers of fully efficient solar cells would be required to meet this demand? The total solar flux at the Earth can be estimated from the solar radius and the radius of the Earth's orbit.

Temp	*R/R_{293K}*	*Resistivity*	*Temp*	*R/R_{293K}*	*Resistivity*
K		*$\mu\Omega$-cm*	*K*		*$\mu\Omega$-cm*
293	1.00	5.48			
300	1.03	5.65	2000	10.34	56.67
400	1.47	8.06	2100	10.96	60.06
500	1.93	10.56	2200	11.58	63.48
600	2.41	13.23	2300	12.21	66.91
700	2.94	16.09	2400	12.84	70.39
800	3.47	19.00	2500	13.49	73.91
900	4.00	21.94	2600	14.14	77.49
1000	4.55	24.93	2700	14.79	81.04
1100	5.10	27.94	2800	15.46	84.70
1200	5.65	30.98	2900	16.12	88.33
1300	6.22	34.08	3000	16.80	92.04
1400	6.79	37.19	3100	17.47	95.76
1500	7.36	40.36	3200	18.16	99.54
1600	7.95	43.55	3300	18.85	103.30
1700	8.54	46.78	3400	19.56	107.20
1800	9.13	50.05	3500	20.27	111.10
1900	9.74	53.35	3600	20.99	115.00

Table II. Resistivity for tungsten as a function of temperature. (From PASCO Web site.)

Color	*λ (nm)*	*Δλ (nm)*	*Trans-mittance*	*f (Hertz)*	*Δf (Hertz)*	*E=hf (eV)*
blue	450	40.1	0.690	6.662×10^{14}	5.945×10^{13}	2.755
green	550	40.1	0.748	5.451×10^{14}	3.982×10^{13}	2.254
red	650	35.8	0.828	4.612×10^{14}	2.542×10^{13}	1.907
deep red	750	40.0	0.500	3.997×10^{14}	2.133×10^{13}	1.653
near IR	850	40.0	0.450	3.527×10^{14}	1.661×10^{13}	1.459
IR	950	40.0	0.450	3.156×10^{14}	1.329×10^{13}	1.305

Table III. Interference filter parameters

λ (nm)	*quantum efficiency*
300	0.413
350	0.504
400	0.547
450	0.646
500	0.748
550	0.824
600	0.846
650	0.852
700	0.855
750	0.852
800	0.843
850	0.832
900	0.814
950	0.783
1000	0.709
1050	0.450
1100	0.180

Table IV: Hamamatsu S2386-5K photodiode quantum efficiency

λ	m_0
950	0.20
850	0.24
750	0.35
650	0.40
550	0.51
450	0.67

Table V. Atmospheric extinction estimated for Ann Arbor

Appendix A

total energy rate $\dot{E} = \frac{2\pi k^4 T^4}{h^3 c^2} \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{2\pi k^4 T^4}{h^3 c^2} \frac{\pi^4}{15}$

total number rate $\dot{N} = \frac{2\pi k^3 T^3}{h^3 c^2} \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{2\pi k^3 T^3}{h^3 c^2} 2\zeta(3)$

mean energy $\langle E \rangle = 2.70117803\ kT = \frac{\pi^4}{30\zeta(3)}\ kT$

energy variance $\sigma_E^2 = 3.05517137\ k^2 T^2 = \left(\frac{12\zeta(5)}{\zeta(3)} - \left(\frac{\pi^4}{30\zeta(3)} \right)^2 \right) k^2 T^2$

energy standard deviation $\sigma_E = 1.74790485\ kT = \sqrt{\sigma_E^2}$

median energy $E_{1/2} = 2.35676306\ kT;\ \int_0^{x_{1/2}} \frac{x^2 dx}{e^x - 1} = \zeta(3)$

first quartile $E_{1/4} = 1.40001979\ kT;\ \int_0^{x_{1/4}} \frac{x^2 dx}{e^x - 1} = \frac{1}{2}\ \zeta(3)$

third quartile $E_{3/4} = 3.63027042\ kT;\ \int_0^{x_{3/4}} \frac{x^2 dx}{e^x - 1} = \frac{3}{2}\ \zeta(3)$

peak $\frac{dI}{df}$ $x_{\max,f} = 2.82143937$; $3 = (3 - x)e^x$

peak $\frac{dI}{d\lambda}$ $x_{\max,\lambda} = 4.96511423$; $5 = (5 - x)e^x$

mean wavelength $\langle \lambda \rangle = 0.68421639\ \frac{hc}{kT} = \frac{\pi^2}{12\zeta(3)} \frac{hc}{kT}$

optimal solar cell bandgap $E_{gap} = 2.16574565\ kT$